IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Extreme-value statistics of 2D chaotic systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1998 J. Phys. A: Math. Gen. 31 8465

(http://iopscience.iop.org/0305-4470/31/42/006)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.104
The article was downloaded on 02/06/2010 at 07:17

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **31** (1998) 8465–8472. Printed in the UK PII: S0305-4470(98)89017-3

Extreme-value statistics of 2D chaotic systems

M C Valsakumar, S V M Satyanarayana and S Kanmani
Materials Science Division, Indira Gandhi Centre for Atomic Research, Kalpakkam, 603102 Tamil Nadu, India

Received 4 November 1997

Abstract. We study the extreme-value statistics of 2D chaotic systems. We consider the extreme value of n phase space points of a 2D chaotic trajectory under a suitably defined norm to order the points and calculate analytically its density. We find that the extreme-value density is non-differentiable on a set of points and the number of such singular points increases with n. However, for identically distributed independent 2D random variables the number of singular points is independent of n.

1. Introduction

Extreme-value statistics is the study of the distribution of $M_n = \max(X_0, X_1, \ldots, X_{n-1})$ where $\{X_0, X_1, \ldots, X_{n-1}\}$ is an n-point sequence. If $F(x)$ is the cumulative distribution of X,

$$F(x) = \mathrm{Prob}(X \leqslant x) \qquad a \leqslant X \leqslant b \tag{1}$$

then the cumulative distribution of M_n is given by

$$F_n(x) = \mathrm{Prob}(X_0 \leqslant x, X_1 \leqslant x, \ldots, X_{n-1} \leqslant x). \tag{2}$$

If $\{X_0, \ldots, X_{n-1}\}$ are independent and identically distributed random variables, then $F_n(x) = (F(x))^n$.

However, the limiting distribution ($n \to \infty$) could be degenerate (i.e. distributed on a single point) unless the extreme values M_n, themselves are appropriately scaled. That is, the limiting distribution of $[a_n M_n - b_n]$ for some suitably chosen sequence of a_n, b_n is non-degenerate. Surprisingly there exists only three classes of limiting distributions [1], namely

$$\begin{aligned} G_{1,\gamma}(x) &= \mathrm{e}^{-(-x)^{\gamma}} 1_{(-\infty,0]}(x) + 1_{[0,\infty)}(x) \\ G_{2,\gamma}(x) &= \mathrm{e}^{-x^{\gamma}} 1_{[0,\infty)}(x) \\ G_{3,\gamma}(x) &= \mathrm{e}^{-\mathrm{e}^{-x}} 1_{(-\infty,\infty)}(x) \end{aligned} \tag{3}$$

where $\gamma > 0$ and the indicator function $1_A(x)$ is defined as

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases} \tag{4}$$

This is due to the fact that extreme-value statistics is determined mainly by the tail of the density of $\{X_i\}$, and the variations in the asymptotic behaviour of the densities are rather restricted. This is in a sense similar to central limit theorem (CLT), where different densities,

0305-4470/98/428465+08$19.50 © 1998 IOP Publishing Ltd

converge to a Gaussian density as a sum. This convergence is also decided by the tail part of the density of the summands. In fact the necessary condition for the CLT is expressed using the notion of regular variation at infinity [2], which also enters the extreme-value statistics in a natural way.

The theory of extreme values was developed in [3–5]. For a recent review on this subject, see [1]. Extreme events/values are important in many areas of physical and applied sciences. For example, the breaking strength of a specimen is determined by its weakest element. Flood is the maximum discharge of water from a river. Extreme behaviour is also of interest in economics. For example, extreme yields may characterize the occurence of bankruptcy or foreign exchange realignments. More recently, it has been applied to diffusion process and economic modelling [6, 7].

The application of extreme-value statistics to 1D chaotic maps has been discussed in [8]. In this work, they studied the distribution of the largest value of the iterates of a 1D chaotic map. They showed that the density of such a variate is discontinuous on a set of points belonging to the unstable periodic orbits of the map. In contrast, they showed that the corresponding probability density for the largest value of the random processes is smooth. Hence, it was pointed out that the deterministic nature of a chaotic process manifests itself as discontinuities in the extreme-value density.

In this paper, we apply the theory of extreme values to 2D chaotic systems. The motivation comes from the following considerations. The coexistence of periodic points of different order (length of the period) and nature (attractive/repulsive) depends not only on the class of the map, but also on the dimension of the space. For example, a diffeomorphism in one dimension cannot have periodic points of order three or more, whereas in $\mathcal{R}^2$, it can have periodic orbits of any order. It would be of interest to investigate the bearing of such differences on the extreme-value statistics. Moreover, since there is no natural ordering in higher dimensional spaces, one has to choose a norm for such order statistics. In principle one can define many norms in a higher dimensional space. However, we choose in our calculations, the norm defined as,

$$r = |x| + |y| = x + y \qquad \text{since } (x, y) \in [0, 1] \times [0, 1] \tag{5}$$

for analytical convenience and compare the results with those obtained by using the usual Euclidean norm. Let $\rho_n(r)$ denote the density of the extreme value of n 2D phase space points, ordered under the norm defined by equation (5). We show that the extreme-value density $\rho_n(r)$ for 2D chaotic systems is non-differentiable on a set of points and the cardinality of this set increases with n. The extreme-value density of 2D random process is also non-differentiable on a set of points. However, the cardinality of the set is independent of n, which distinguishes chaos from random processes.

2. 2D random process

Let $\{X_i\}$ be independent and identically distributed random variables in $[0,1]^d$, i.e. $X = (x_1, x_2, \ldots, x_d)$ is a d-dimensional vector. Let the norm be that of (5)

$$r = \|X\| = \sum_{i=1}^{d} x_i. \tag{6}$$

If $P(r)$ is the probability density of r and $\xi_l = \max(r_1, r_2, \ldots, r_l)$ is the extreme value, then the cumulative distribution of ξ_l is given by

$$F_l(\xi) = \text{Prob}(\xi_l \leqslant \xi). \tag{7}$$

One can show if $P(r) \in C^m$, then $F_l(\xi) \in C^{m+1}$. The extreme-value distribution with respect to the given norm is

$$F_l(\xi) = [F(\xi)]^l \qquad l \geqslant 2 \tag{8}$$

where

$$F(\xi) = \int_0^\xi P(r)\,\mathrm{d}r. \tag{9}$$

The derivative of $F_l(\xi)$ gives the extreme-value density

$$\rho_l(\xi) = \frac{\mathrm{d}F_l(\xi)}{\mathrm{d}\xi} = l[F(\xi)]^{l-1}P(\xi). \tag{10}$$

Consider the $(m+1)$th derivative of $F_l(\xi)$,

$$\frac{\mathrm{d}^{m+1}}{\mathrm{d}\xi^{m+1}}F_l(\xi) = l\sum_{j=0}^{m-1} {}^mC_j \frac{\mathrm{d}^{m-j}}{\mathrm{d}\xi^{m-j}}[F(\xi)]^{l-1}\frac{\mathrm{d}^j}{\mathrm{d}\xi^j}P(\xi). \tag{11}$$

This involves only derivatives up to order m of $P(\xi)$, which implies if $P(\xi) \in C^m$, the r.h.s. of equation (11) is finite and $F_l(\xi) \in C^{m+1}$.

Thus, it is clear from (10) that if the invariant density of higher dimensional random process is non-differentiable on a set of points, the extreme-value density is also non-differentiable on the same set of points. In contrast, the extreme-value density of 1D random process is smooth. It also follows from (10) that the number of non-differentiable points in the extreme-value density is independent of length of the data set. For example, the $\rho_n(r)$ of 2D random process with the invariant density of the components being uniform, can be calculated exactly as

$$\rho_n(r) = \begin{cases} \dfrac{n}{2^{n-1}}r^{2n-1} & 0 \leqslant r < 1 \\ \dfrac{n}{2^n}(4r - r^2 - 2)^{n-1}(4 - 2r) & 1 \leqslant r < 2. \end{cases} \tag{12}$$

Figure 1 shows $\mathrm{d}\rho_n(r)/\mathrm{d}r$ for $n = 2$, 3 and 5. The $\rho_n(r)$ of 2D random vectors is non-differentiable at $r = 1$ for all n.

3. 2D chaotic systems

Consider a 2D sequence $\boldsymbol{X}_n = (x_n, y_n)$ of points generated by the map

$$\begin{aligned} x_{n+1} &= f_x(x_n, y_n) \\ y_{n+1} &= f_y(x_n, y_n). \end{aligned} \tag{13}$$

Let $\rho(x, y)$ be the invariant distribution of the system. The joint probability of the first n values of the map with $y_i = r_i - x_i$, is given by

$$\rho((x_1, r_1 - x_1), \ldots, (x_n, r_n - x_n)) = \int \mathrm{d}x_0 \int \mathrm{d}y_0\, \rho(x_0, y_0) \times \prod_{m=1}^{n} \delta(x_m - f_x^{(m)}(x_0, y_0))\delta(r_m - x_m - f_y^{(m)}(x_0, y_0)) \tag{14}$$

where $(f_x^{(m)}(x_0, y_0), f_y^{(m)}(x_0, y_0))$ is the mth iterate of (x_0, y_0).

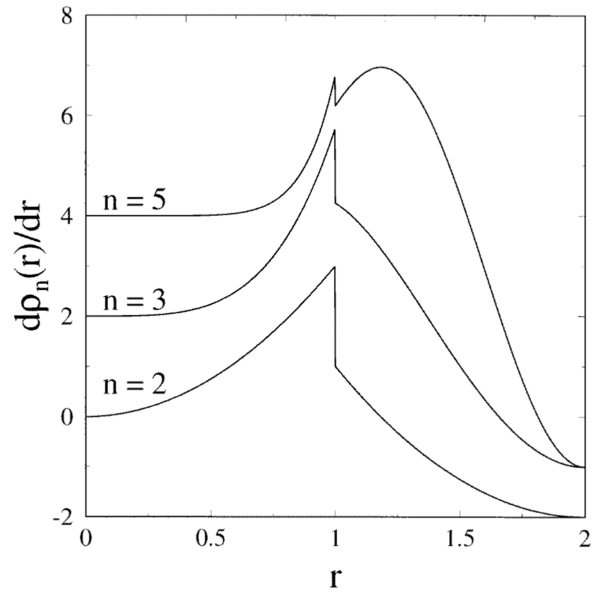

Figure 1. The derivative of the extreme-value density of a 2D random process with respect to a norm defined in equation (5), $\mathrm{d}\rho_n(r)/\mathrm{d}r$ (see equation (12)).

The cumulative distribution of the extreme value is given by

$$F_n(r) = \mathrm{Prob}(r_0 \leqslant r, r_1 \leqslant r, \ldots, r_n \leqslant r)$$
$$= \int \mathrm{d}x_0 \int \mathrm{d}y_0\, \rho(x_0, y_0) \prod_{m=1}^{n} \Theta(r - f_x^{(m)}(x_0, y_0) - f_y^{(m)}(x_0, y_0)). \tag{15}$$

We calculate in the following sections, $F_n(r)$ for 2D maps with uniform invariant density. The analytical expressions for $n = 2$, is obtained where the integration is carried out using a geometric construction.

The density function $\rho_n(r)$ of the extreme values with respect to the norm defined is obtained by differentiating $F_n(r)$

$$\rho_n(r) = \int \mathrm{d}x_0 \int \mathrm{d}y_0\, \rho(x_0, y_0) \sum_{m=1}^{n} \prod_{k \neq m} \delta(r - f_x^{(m)}(x_0, y_0) - f_y^{(m)}(x_0, y_0))$$
$$\times \Theta(r - f_x^{(k)}(x_0, y_0) - f_y^{(k)}(x_0, y_0)). \tag{16}$$

In general, the graph, $f_x^{(m)}(x_0, y_0) + f_y^{(m)}(x_0, y_0) = r_\star$ in a unit square can be discontinuous either at a point or across a line, depending on the map defined. If the above line is discontinuous at a point, then the discontinuity gets smoothened by integration and only the derivative of $\rho_n(r)$ shows a discontinuity at $r = r_\star$. If the discontinuity is across a line, then $\rho_n(r)$ itself will be discontinuous at $r = r_\star$. We illustrate these features below with examples.

The equation similar to (16) derived for 1D chaotic systems has contributions to the integral when $f^{(m)}(x_0) = x_0$, which are the periodic points of the map. But in 2D, the equation $f_x^{(m)}(x_0, y_0) + f_y^{(m)}(x_0, y_0) = r$ does not correspond to a point. Instead it corresponds to a line with respect to the chosen norm (5), or corresponds to a circle in the case of Euclidean norm. Thus, the points of non-differentiability cannot be related to the periodic points of the system.

We present below the results obtained for two model chaotic maps in two dimensions [9, 10], namely

$$(x_{n+1}, y_{n+1}) = \begin{cases} \left(2x_n, \dfrac{y_n}{2}\right) & \text{if } 0 \leqslant x_n < \frac{1}{2} \\ \left(2x_n - 1, \dfrac{y_n + 1}{2}\right) & \text{if } \frac{1}{2} \leqslant x_n < 1 \end{cases} \tag{17}$$

known as Baker's map and

$$(x_{n+1}, y_{n+1}) = \begin{cases} (x_n + y_n, x_n) & \text{if } 0 \leqslant y_n < 1 - x_n \\ (x_n + y_n - 1, x_n) & \text{if } 1 - x_n \leqslant y_n < 1. \end{cases} \tag{18}$$

The expression for cumulative probability distribution of the extreme value of a 2-point data set is given by

$$F_2(r) = \int \mathrm{d}x_0 \int \mathrm{d}y_0\, \rho(x_0, y_0)\Theta(r - f_x(x_0, y_0) - f_y(x_0, y_0)) \\ \times\Theta(r - f_x^2(x_0, y_0) - f_y^2(x_0, y_0)). \tag{19}$$

As the invariant density $\rho(x_0, y_0)$ is uniform the above integral can be simplified by substituting for f_x and f_y in different ranges.

$$\begin{aligned} F_2(r) = & \int_0^{\frac{1}{4}} \mathrm{d}x_0 \int_0^1 \mathrm{d}y_0\, \Theta\left(r - 2x_0 - \frac{y_0}{2}\right) \Theta\left(r - 4x_0 - \frac{y_0}{4}\right) \\ & + \int_{\frac{1}{4}}^{\frac{1}{2}} \mathrm{d}x_0 \int_0^1 \mathrm{d}y_0\, \Theta\left(r - 2x_0 - \frac{y_0}{2}\right) \Theta\left(r - 4x_0 + 1 - \frac{y_0}{4} - \frac{1}{2}\right) \\ & + \int_{\frac{1}{2}}^{\frac{3}{4}} \mathrm{d}x_0 \int_0^1 \mathrm{d}y_0\, \Theta\left(r - 2x_0 + 1 - \frac{y_0}{2} - \frac{1}{2}\right) \Theta\left(r - 4x_0 + 2 - \frac{y_0}{4} - \frac{1}{4}\right) \\ & + \int_{\frac{3}{4}}^{1} \mathrm{d}x_0 \int_0^1 \mathrm{d}y_0\, \Theta\left(r - 2x_0 + 1 - \frac{y_0}{2} - \frac{1}{2}\right) \Theta\left(r - 4x_0 + 3 - \frac{y_0}{4} - \frac{3}{4}\right). \end{aligned} \tag{20}$$

Consider the first term in equation (20) and the integral in the range $0 \leqslant r < 0.5$ is the area of ABCD in the figure 2, where line 1 and line 2 are $2x_0 + y_0/2 = r$ and $4x_0 + y_0/4 = r$, respectively.

$F_2(r)$ is obtained by integrating each of the term in (20) in different ranges of r as described above. $\rho_2(r)$ which is the derivative of $F_2(r)$ is given by

$$\rho_2(r) = \begin{cases} \dfrac{2r}{3} & 0 \leqslant r < \dfrac{1}{2} \\ \dfrac{4r}{3} - \dfrac{1}{3} & \dfrac{1}{2} \leqslant r < 1 \\ -\dfrac{2r}{3} + \dfrac{5}{3} & 1 \leqslant r < \dfrac{3}{2} \\ -\dfrac{4r}{3} + \dfrac{8}{3} & \dfrac{3}{2} \leqslant r < 2. \end{cases} \tag{21}$$

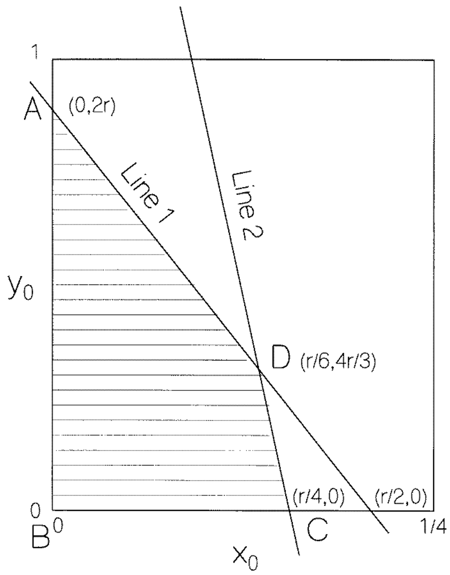

Figure 2. Diagram illustrating the evaluation of integral in the first term of equation (23) in the range $0 \leqslant r < 0.5$.

A similar exercise has been carried out for the map defined in equation (18) and the corresponding $\rho_2(r)$ is

$$\rho_2(r) = \begin{cases} \dfrac{r}{2} & 0 \leqslant r < 1 \\ -\dfrac{3r}{2} + 3 & 1 \leqslant r < 2. \end{cases} \tag{22}$$

The extreme-value density of a 2D chaotic map $\rho_n(r)$ with respect to a defined norm r can also be numerically computed. Given a map with invariant density (mere existence is necessary and sufficient to render meaning for the averaging procedure), the extreme-value density can be computed in the following way. Starting from an initial condition, n iterates (each being a 2D vector) of the map are obtained. These iterates are ordered based on a norm and the maximum is thus picked up. This is repeated for several initial conditions to obtain a histogram representing $\rho_n(r)$. The typical number of initial conditions used in the computation of $\rho_n(r)$ is 10^7. Numerically computed $\rho_n(r)$, $n = 2$, $n = 3$, $n = 5$ of Baker's map defined in (17) is shown in figure 3. $\rho_2(r)$ in figure 3 agrees with the analytical result (21). Also, it can be seen that the number of non-analytic points of $\rho_n(r)$ increases as n increases. In contrast they remain constant in random processes, see figure 1. Thus, the feature that distinguishes chaos from random process in higher dimension is the *increase* in the number of non-differentiable points of $\rho_n(r)$ with n.

4. Discussion

The points of non-differentiability of $\rho_n(r)$ of Baker's map (17) are $r = \frac{1}{2}$, 1 and $\frac{3}{2}$. Baker's map has two fixed points given by $(0, 0)$ and $(1, 1)$. The period-two points are $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$. The values of norm corresponding to the above periodic points are $r = 0$, 1 and 2. It is clear that these points do not coincide with the points of non-differentiability of $\rho_2(r)$ of Baker's map. This is in contrast with the 1D case where the corresponding points belong to the periodic orbits.

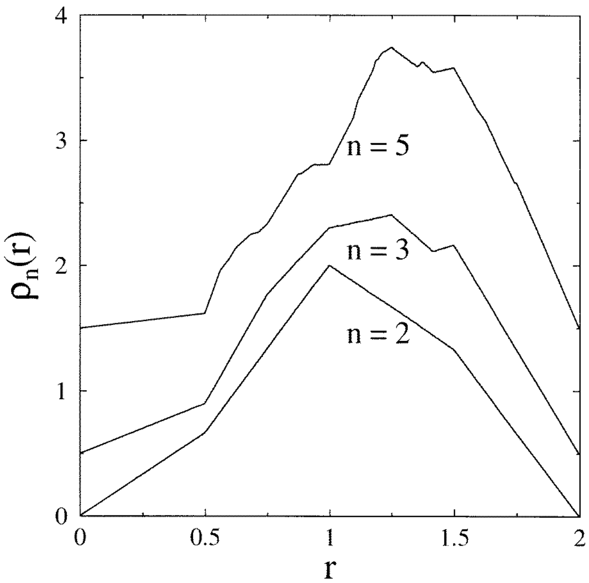

Figure 3. Extreme-value density $\rho_n(r)$ of Baker's map with respect to norm defined in equation (5). $\rho_n(r)$ is non-differentiable on the set of points. Note that these points increase with n (see equation (21) for the analytical expression).

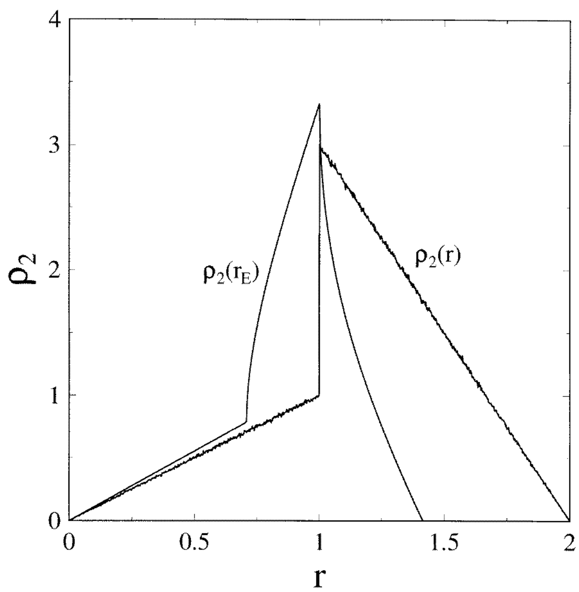

Figure 4. Extreme-value density ρ_2 of the map defined in equation (18) with respect to norm defined in equation (5) and Euclidean norm (see equation (22) for analytical expression).

$\rho_2(r)$ of the map defined in (18) is shown in figure 4. $\rho_2(r)$ is discontinuous at $r = 1$. One can show that the line $f_x^{(2)}(x_0, y_0) + f_y^{(2)}(x_0, y_0) = 1$ is discontinuous across the lines $2x_0 + y_0 = 1$, $x_0 + y_0 = 1$ and $2x_0 + y_0 = 2$.

We numerically investigate the extreme-value statistics of 2D maps with respect to

Euclidean norm, $r_E = \sqrt{x^2 + y^2}$. $\rho_n(r_E)$ is also non-differentiable on a set of points. These points also increase with n and hence distinguish chaos from a random process. However, the points of non-differentiability of $\rho_n(r_E)$ do not have one-to-one correspondence with the non-differentiable points of $\rho_n(r)$. For example, $\rho_2(r_E)$ of the second system (18) shows that $\rho_2(r_E)$ is non-analytic at two points, but has no discontinuity. In contrast, the $\rho_2(r)$ of the same map is discontinuous at $r = 1$ which is the only non-differentiable point, see figure 4.

5. Summary

In summary, the extreme-value density of 2D chaotic systems is calculated with respect to a norm. $\rho_n(r)$ is non-differentiable on a set of points which do not belong to the periodic orbits of the system. However, we believe that these points are related to the unstable periodic orbits in some non-trivial fashion. It would be of interest to find out such a relation through some variant of this analysis. The number of such non-differentiable points of $\rho_n(r)$ increases with n. In contrast, the number of points of non-differentiability of two- and higher-dimensional random processes does not increase with n. This feature in the extreme-value density distinguishes chaos from random processes in higher dimensions.

Acknowledgments

We thank K P N Murthy for his suggestions to improve the readability of the manuscript. SVMS acknowledges CSIR, India for the award of Senior Research Fellowship.

References

[1] Leadbetter M R and Rootzen H 1988 *Ann. Prob.* **16** 431
[2] Port S C 1994 *Theoretical Probability for Applications* (New York: Wiley)
[3] Fisher R A and Tippet L H C 1928 *Proc. Camb. Phil. Soc.* **24** 180
[4] Gnedenko B V 1943 *Ann. Math.* **44** 423
[5] Gumbel E J 1958 *Statistics of Extremes* (New York: Columbia University Press)
[6] Kehr K W, Murthy K P N and Ambaye H 1998 *Physica* **253A** 9
[7] de Haan L, Resnick S I, Rootzen H and de Vries C 1988 Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes *Technical Report* no 233, Department of Statistics, University of North Carolina, NC, USA
[8] Balakrishnan V, Nicolis C and Nicolis G 1995 *J. Stat. Phys.* **80** 307
[9] Devaney R L 1986 *An Introduction to Chaotic Dynamical Systems* (New York: Benjamin)
[10] Lasota A and Mackey M C 1994 *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics* (Berlin: Springer)